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The Role of Coding Time in Estimating and Interpreting Growth Curve Models

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The coding of time in growth curve models has important implications for the interpretation of the resulting model that are sometimes not transparent. The authors develop a general framework that includes predictors of growth curve components to illustrate how parameter estimates and their standard errors are exactly determined as a function of recoding time in growth curve models. Linear and quadratic growth model examples are provided, and the interpretation of estimates given a particular coding of time is illustrated. How and why the precision and statistical power of predictors of lower order growth curve components changes over time is illustrated and discussed. Recommendations include coding time to produce readily interpretable estimates and graphing lower order effects across time with appropriate confidence intervals to help illustrate and understand the growth process.

The analysis of change over time has been the focus of considerable theoretical interest and attention historically (e.g., Gompertz, 1820, 1825; Harris, 1963; Quetelet, 1835/1980; Reed & Pearl, 1927; Tucker, 1958; Wishart, 1938). Recent decades have seen the development of sophisticated quantitative approaches to address questions of stability and change over time (e.g., Collins & Horn, 1991; Jöreskog, 1979; Meredith & Tisak, 1984, 1990; Rogosa, Brandt, & Zimowski, 1982; Tisak & Tisak, 2000). One approach to examining questions of stability and change is the analysis of individual growth curve trajectories. Excellent and accessible didactic articles describe how to estimate growth curve models using either *multilevel models,* also known as *random coefficient* or *hierarchical linear models,* or *structural equation models* (SEMs; e.g., Bryk & Raudenbush, 1987; Chou, Bentler, & Pentz, 1998; MacCallum, Kim, Malarkey, & Kiecolt-Glaser, 1997; Raudenbush & Bryk, 2002; Singer, 1998; Willett & Sayer, 1994).

As welcome as the development of sophisticated techniques for addressing questions of stability and change has been, the practical utility of statistical models is limited by the ability of researchers to appropriately analyze data and draw substantive conclusions. Of particular concern for growth curve models are the ramifications of how time is coded for the estimation and interpretation of growth curve trajectory parameters. The past several years have seen researchers publishing articles and chapters that contain incorrect interpretations of their model parameters. A common misinterpretation is the meaning of the intercept given a particular coding of time and specified functional form of growth (e.g., linear, quadratic, cubic). For example, Kurdek (1999, pp. 1283, 1287) and

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Windle (1997, pp. 57–59) both have interpreted the intercept as initial status when, according to how they have coded time in their models, it was actually the average expected value across assessments.

Misinterpretations or concerns regarding the relationships among growth parameters (e.g., intercepts and slopes) across individuals appear frequently as well. Covariances among growth parameters across individuals have been interpreted as (a) indicative of multicollinearity and thus potentially problematic (e.g., Smith, Landry, & Swank, 2000, p. 38), (b) regression to the mean (Wickrama, Lorenz, & Conger, 1997, p. 156), or (c) apparently not even modeled and thus implicitly constrained to zero (e.g., Barnes, Reifman, Farrell, & Dintcheff, 2000, p. 181). Because, as we demonstrate in this article, these covariances are directly determined by the choice of how to code time, the common thread linking these examples is the need to know the direct impact of different codings of time on parameter estimates and, consequently, the interpretation of these estimates. It is possible to equivalently reformulate and reexpress these models to achieve these researchers' desired interpretations as well as many other possibilities simply by recoding time. Although the choice of how to code and scale time may at times appear arbitrary, the interpretation of the resulting solution is not.

In growth curve models, as their name implies, change on a variable (growth) is modeled as a function of time. Within the context of multiple regression, the impact of predictor scaling choices on parameter estimates and their interpretation has received considerable attention (e.g., Aiken & West, 1991; Belsley, 1984; Cohen, 1978; Marquardt, 1980; Snee & Marquardt, 1984; Wainer, 2000; West, Aiken, & Krull, 1996), but little work has been done in the more complex case of growth curve models.[1]

Notable exceptions to this include the groundbreaking work of Rogosa and Willett (1985; see also Rogosa, Brandt, & Zimowski, 1982), who derived exact expressions for the impact of recoding time on the relationship between the intercept and linear components of linear growth curve models. In the multilevel modeling tradition, Raudenbush and colleagues have repeatedly emphasized parameterizing growth curve models to address the specific substantive questions of import to the researcher (Raudenbush, 2001a; 2001b; Raudenbush & Bryk, 2002). In addition, Schuster and von Eye (1998) recently argued that, just as in multiple regression, coding choices for growth curve models are critical for producing interpretable parameter estimates (see also Kreft, de Leeuw, & Aiken, 1995; Mehta & West, 2000; Muthén, 2000). However, there has not been a systematic treatment of the impact of different choices of coding time on all parameters and corresponding standard errors of a growth curve model that includes predictors of growth curve components.

The purpose of this article is thus to demonstrate analytically how growth curve parameters and their standard errors change in a deterministic manner under different codings or parameterizations of time and to illustrate the substantive and interpretative impact of these changes. First we briefly review the interpretation of regression parameters and their relationship to growth curve models. Next, through an empirical illustration using linear and quadratic growth models, we present general analytic results demonstrating how to determine different parameter estimates as well as their standard errors in growth curve models as a function of recoding time and the interpretation of those parameter estimates. Using the quadratic growth model as an example, we examine and discuss centering time and orthogonal polynomial contrast codes as default coding strategies. However, we encourage researchers not to adopt these default strategies and instead to use the formulas developed in this article to graph lower order effects in growth curve models over time with appropriate confidence intervals (CIs) in order to provide new insights and link the graphical presentation of the growth process to their original specific substantive questions. Finally, we illustrate and discuss how the precision of predictors of the growth process and their statistical power change as a function of recoding time and depend in large part on the underlying growth process.

Coding Time and Interpreting Parameter Estimates in Growth Curve Models

Consider the one-predictor regression equation where person i's observed value on a variable of

[1] Mehta and West (2000) recently emphasized the importance of appropriately and precisely measuring time in growth curves. Noting that relationships among linear growth curve components change when the scale of time changes, if individuals are measured at different points in time, but coded similarly, biases in estimation can result. Using a time-balanced design for illustrative purposes, we develop general formulas for understanding the different solutions that result given different codings of time given a precise measurement of time.

interest (y_i) is modeled as a linear function of time (t):

$$y_{it} = \eta_{0i} + \eta_{1i}t_{it} + \varepsilon_{it}. \tag{1}$$

This represents a linear growth curve model for a single individual. The regression coefficient η_{1i} is the expected change in y for a 1-unit change in time for person i. The intercept η_{0i} is the expected value of y at the origin of time (i.e., when t equals 0). Thus the choice of where to place the origin of time has a direct and predictable effect on the estimate and interpretation of the intercept for person i.

Interpreting regression coefficients for a single regression equation as in Equation 1 is straightforward. However, growth curve models estimate growth curves for many individuals simultaneously. This involves moving from considering an isolated regression equation for a single individual to examining the relationship of observations across individuals. As an illustration, consider a set of individuals, each of whom has a growth curve as in Equation 1 and, without loss of generality, shares the same assessment schedule. The covariance matrix and mean of observations across individuals across time can be expressed as follows:[2]

$$\Sigma_{yy} = \boldsymbol{\Lambda}_i\boldsymbol{\Psi}\boldsymbol{\Lambda}_i' + \boldsymbol{\Theta}_{\varepsilon\varepsilon},$$

and

$$\boldsymbol{\mu}_y = \boldsymbol{\Lambda}_i\boldsymbol{\mu}_\eta, \tag{2a}$$

where

$$\boldsymbol{\Lambda}_i = \begin{bmatrix} 1 & t_{i0} \\ 1 & t_{i1} \\ 1 & t_{i2} \\ \vdots & \vdots \\ 1 & t_{ic} \end{bmatrix}. \tag{2b}$$

The loading matrix $\boldsymbol{\Lambda}_i$ specifies the coding of time as well as the functional form of the individual growth curves over the $c + 1$ repeated observations which, in this case, is linear. The two columns of $\boldsymbol{\Lambda}_i$ correspond to the intercept and linear components of time, respectively. Observations may or may not be made at equally spaced intervals. Estimates of the variances of intercept and linear trajectory components across individuals as well as their covariance are contained in the $\boldsymbol{\Psi}$ matrix. Uniquenesses associated with each observed variable are contained in the $\boldsymbol{\Theta}_{\varepsilon\varepsilon}$ matrix. The mean of the regression coefficients across individuals is estimated and provided in the vector $\boldsymbol{\mu}_\eta$.

Examining growth curves across individuals simultaneously allows us to ask questions of what factors or variables predict growth curve components—a conditional growth curve model. If predictors of the growth curve components across individuals are centered without loss of generality and included in the model, the equations for predicting the intercept and slope across individuals are as follows:

$$\eta_{0i} = \boldsymbol{\mu}_{\eta 0} + \boldsymbol{\Gamma}_0 X_i + \boldsymbol{\zeta}_{0i},$$

$$\eta_{1i} = \boldsymbol{\mu}_{\eta 1} + \boldsymbol{\Gamma}_1 X_i + \boldsymbol{\zeta}_{1i}. \tag{3a}$$

In this conditional growth model, the covariance matrix of observations across individuals across time can be expressed as follows:

$$\Sigma_{yy} = \boldsymbol{\Lambda}_i(\boldsymbol{\Gamma\Phi\Gamma}' + \boldsymbol{\Psi})\boldsymbol{\Lambda}'_i + \boldsymbol{\Theta}_{\varepsilon\varepsilon}. \tag{3b}$$

The matrix $\boldsymbol{\Gamma}$ contains the regression weights of predictors of the growth curve components across individuals, and the $\boldsymbol{\Phi}$ matrix is the covariance matrix of the centered exogenous predictors X. In the conditional growth curve model, the $\boldsymbol{\Psi}$ matrix represents the covariance matrix of the $\boldsymbol{\zeta}$s—residual growth curve variability across individuals unaccounted for by the predictors X. The vector $\boldsymbol{\mu}_\eta$ is interpreted as an intercept—the predicted growth curve when the set of predictors X are all equal to 0—which is the mean for centered predictors.

The move from considering a single individual's growth curve regression equation as in Equation 1 to examining multiple individuals' growth curves simultaneously raises questions of interpretation. How does recoding time affect estimates across individuals and, consequently, their interpretation? For the linear growth curve model, Mehta and West (2000) demonstrated how the choice of placing time's intercept impacts the variance of the intercept and the covariance between intercept and slope across individuals (for a

[2] These equations represent the SEM approach to growth curve models. Several excellent articles illustrate how, under some general conditions, the multilevel model and the SEM approaches are equivalent formulations of the same model (e.g., MacCallum, Kim, Malarkey, & Kiecolt-Glaser, 1997; Willett & Sayer, 1994; see also Raudenbush & Bryk, 2002). For ease of presentation, we focus primarily on the SEM formulation, but note that all results and discussions are equally applicable to the multilevel model and unbalanced data.

more detailed discussion, see Rogosa & Willett, 1985). For growth curve models, the choice of how to code the predictors (i.e., time) determines not only the meaning of the regression parameters for each individual but also how these parameters vary across individuals and how to interpret that interindividual variability. To illustrate the determination and interpretation of growth curve model parameters as a function of recoding time, we first examine a linear and then quadratic empirical growth curve model and, subsequently, extend this example to the conditional quadratic growth curve model by including a predictor of the growth curve parameters.

Empirical Illustrations

The linear growth curve model. As an illustration, we first examine changes in children's weight. The means, standard deviations, and correlation matrix for the weight in pounds (1 lb = 0.45 kg) of 155 children at ages 5, 7, 9, 11, and 13 obtained from the National Longitudinal Survey of Youth (NLSY; Baker, Keck, Mott, & Quinlan, 1993) are presented in Table 1 and are the basis for all presented analyses. First, suppose we are interested in children's initial weight at age 5 and its relationship with the rate of change in weight growth from ages 5 to 9. Note that we first use a linear growth curve model to examine ages 5 to 9 and later use a quadratic model to examine the wider age range of 5 to 13. To obtain estimates of children's weight at age 5, we place the origin of time at age 5 by coding time = (age – 5), which results in the following loading matrix for all individuals for the first three assessments:

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}.$$

To preserve years as the unit of time, the linear component of the $\boldsymbol{\Lambda}$ matrix increases in increments of 2, as assessments of the children's weight were made every 2 years. The parameter estimates for this linear growth model are presented in Table 2 under Model A and represent the data well, $T_{\mathrm{ML}}(1, N = 155) = 2.13$, *ns,* comparative fit index (CFI) = .999, incremental fit index (IFI) = .999, Tucker–Lewis Index (TLI) = .996, root-mean-square error of the approximation (RMSEA) = .086, 90% CI = .000–.250. Note that the model test statistic T_{ML} is asymptotically distributed as a chi-square when the model assumptions are met (see Curran, Bollen, Paxton, Kirby, & Chen, 2002). All models were estimated using maximum likelihood in Amos (Version 4.01; Arbuckle, 1999). The mean intercept value of 39.457 indicates the average weight of the sample of children at age 5. The variance for the intercept of 28.776 indicates that there was substantial variability across children in their weight at this age. The average linear trajectory across children, 8.063, indicates that children, on average, were gaining just over 8 lbs a year. However, the significant variance among linear slopes indicates substantial individual variability in the rate of weight gain across children. Note that the significant positive covariance between the intercept and linear slope indicates that at age 5 heavier children are gaining weight at a faster rate than children who weigh less.

Now suppose that we are interested in understanding children's weight at a later age—specifically at age 9. To obtain this information, we would place the origin of time at age 9 by coding time = (age – 9). This would result in the following loading matrix:

$$\boldsymbol{\Lambda}^* = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}.$$

Table 1
Intercorrelations Between Children's Weight From Ages 5 to 13 and Mother's Weight Before Pregnancy (N = 155)

Weight	1	2	3	4	5	6
1. Child at age 5	—					
2. Child at age 7	.7947	—				
3. Child at age 9	.7264	.8569	—			
4. Child at age 11	.6405	.7866	.8651	—		
5. Child at age 13	.6025	.7447	.7968	.8981	—	
6. Mother	.1592	.2891	.3550	.4175	.4296	—
M	39.5480	55.3160	72.3350	96.2520	119.1030	127.1670
SD	6.1096	11.1546	17.8567	26.9084	33.4412	21.2030

Note. Weight is given in pounds (1 lb. = 0.45 kg). Children at age 5 were between 55 and 60 months old when assessed in 1988 in the National Longitudinal Survey of Youth.

Table 2
Linear Unconditional Model Estimates and Standard Errors Under Three Time Coding Schemes for Ages 5 to 9

	Model					
	A		B		C	
Parameter	Estimate	*SE*	Estimate	*SE*	Estimate	*SE*
Variance						
Intercept	28.776	6.143	260.032	33.037	111.599	13.522
Linear	8.201	1.470	8.201	1.470	8.201	1.470
Covariance						
Intercept–linear	12.505	2.676	45.309	6.382	28.907	3.912
M						
Intercept	39.457	0.487	71.707	1.369	55.582	0.877
Linear	8.063	0.267	8.063	0.267	8.063	0.267
Unique variance						
Age 5	8.319	5.280	8.319	5.280	8.319	5.280
Age 7	12.094	4.880	12.094	4.880	12.094	4.880
Age 9	57.168	15.992	57.168	15.992	57.168	15.992
Model $T_{ML}(df = 1)$	2.131		2.131		2.131	

Note. The T_{ML} is the model test statistic and is asymptotically distributed as a chi-square when the model assumptions are met. Model loading matrices were as follows for Models A, B, and C, respectively:

$$\boldsymbol{\Lambda}_A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}, \boldsymbol{\Lambda}_B = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}, \boldsymbol{\Lambda}_C = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Parameters for the model with time's origin at age 9 are distinguished from the previous model by the addition of an asterisk. Estimating this model with the intercept placed at age 9 does not alter or change the fundamental growth process observed in the model with the intercept at age 5. By recoding time, we are simply reorganizing or reparameterizing the same information to provide answers to different specific sub-

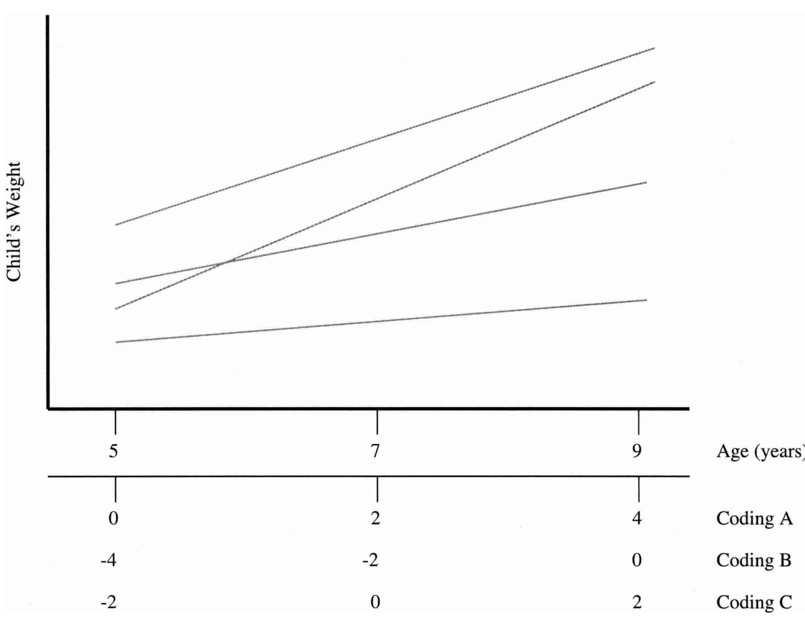

Figure 1. Four hypothetical latent linear trajectories coding age under three different time coding systems. The latent trajectories remain invariant regardless of how age is coded.

stantive questions. This can be seen in Figure 1, which illustrates four hypothetical latent linear trajectories under several different time codings. Although the coding of time and the interpretation of the model's parameters may change, the underlying latent trajectories do not.

Consequently, we can analytically determine the exact solution for a different coding of time, such as the intercept at age 9, as a function of a previous solution. Specifically, we can determine the covariance matrix of the regression coefficients and their means for the intercept placed at age 9 (i.e., $\mathbf{\Psi}^*$ and $\boldsymbol{\mu}_\eta^*$ given $\mathbf{\Lambda}^*$) as a function of the solution when the intercept was placed at age 5 (i.e., $\mathbf{\Psi}$ and $\boldsymbol{\mu}_\eta$ given $\mathbf{\Lambda}$) using a transformation matrix $\mathbf{T}$, as follows (see the Appendix for the derivation):

$$\mathbf{\Psi}^* = \mathbf{T}^{-1}\mathbf{\Psi}\mathbf{T}^{-1\prime}, \tag{4}$$

and

$$\boldsymbol{\mu}_\eta^* = \mathbf{T}^{-1}\boldsymbol{\mu}_\eta, \tag{5}$$

where

$$\mathbf{T}^{-1} = (\mathbf{\Lambda}^{*\prime}\mathbf{\Lambda}^*)^{-1}\ \mathbf{\Lambda}^{*\prime}\mathbf{\Lambda}.$$

Through the use of these two formulas, the solution for the model with the intercept recoded to be at age 9 is readily and exactly obtained without the necessity of separately estimating this new model. For example, using Equation 4, the variance of the intercept at age 9 can be directly determined to be 260.032. For comparison, the solution estimated by Amos 4.01 is provided as Model B in Table 2.

Note that the direct determination of growth curve components under a different coding of time holds even when predictors of the growth curve components are included in the model. For conditional growth curve models—which include predictors of the growth curve components—Equations 4 and 5 remain unchanged, although these components are now interpreted as residual variances and intercepts, respectively. Equation 6 below describes how the relationship between predictors and growth curve components can be exactly determined as a function of recoding time:

$$\mathbf{\Gamma}^* = \mathbf{T}^{-1}\mathbf{\Gamma}. \tag{6}$$

Standard errors for all parameter estimates under a new recoding of time are exactly determined as well. Thus we can determine, for example, that the standard error of the variance of the intercept at age 9 is 33.037 without having to reestimate the model. The analytic derivation of transforming standard errors is presented in the Appendix, with this particular example illustrated in detail.

Parameter estimates for these two models are presented in Table 2. In addition, Model C in Table 2 provides the solution when time is centered within the period of these first three observations—that is, coding time = (age −7). Although all three models, which differ only in how time is coded, can be translated into each other exactly through Equations 4 and 5, it is worth considering the broad information contained by each. These linear growth curve models tell us about individual differences in children's weight at a specific age (i.e., the intercept) as well as the rates of change in weight across time (i.e., the slope). The choice(s) of where to examine individual differences at a specific point in time depends on the specific questions of interest to the researcher. The model can be transformed to examine individual differences at a different point in time. However, this raises a question that requires clarification: Under what conditions do Equations 4–6 hold for different codings of time or other reparameterizations?

The technical requirement, outlined in the Appendix, is that an exact relationship exists between $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^*$ such that $\mathbf{\Lambda}^* = \mathbf{\Lambda}\mathbf{T}$, where $\mathbf{T}$ is a nonsingular square matrix that transforms $\mathbf{\Lambda}$ into $\mathbf{\Lambda}^*$. Any rescaling or recoding of time such that time* $= a + b$(time) where $b \neq 0$ will satisfy this requirement, as it can be specified through such a nonsingular $\mathbf{T}$ transformation matrix. The size of the $\mathbf{T}$ matrix is determined by the shape of the estimated growth function—the order of the $\mathbf{T}$ matrix is the same as the number of growth curve components and thus conforms with the $\mathbf{\Lambda}$ matrix. This can be illustrated with the loading matrices presented in the previous linear example:

$$\mathbf{\Lambda}^* = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \end{bmatrix} = \mathbf{\Lambda}\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

The $\mathbf{T}$ matrix for this particular recoding of time reparameterizes the model—shifting time's origin from age 5 to age 9—and reorganizes the same basic information present in the loading matrices. This is done by simply subtracting 4 from the loadings for the linear component. Note that any transformation, such as recoding time, does not change the model's overall fit as it results in an equivalent model in the sense of MacCallum, Wegener, Uchino, and Fabrigar (1993). That is, the model-implied covariance matrix and the model test statistic (T_{ML}) are identical under these transformations. These transformation equations do not allow moving between different estimated functional forms of growth such as transforming the linear growth model into a quadratic growth model or moving between a linear growth model and an unstructured growth model in which some loadings are estimated (e.g., Meredith & Tisak, 1990). The same functional form of growth (e.g., linear, quadratic) must be held constant, and the **T** transformation matrix ensures that requirement.

Note that the **T** transformation matrix remains constant regardless of when observations are made. For example, recoding the intercept from age 5 to age 9 results in the same transformation matrix, irrespective of the timing of the actual observations. For ease of presentation, we consider in detail a balanced design in which all individuals are assessed at the same time points and there are no missing data. This allows the interested reader to use Table 1 to recreate all analyses presented in this article. However, this is not a requirement or limitation. The difficulty presented with unbalanced and missing data lies with estimating the model. However, once a model is estimated, time can be recoded using the equations developed in this article, as we illustrate briefly later with a fully unbalanced example.[3]

The quadratic growth curve model. We have presented the linear growth model across ages 5 to 9 primarily for expository purposes. Now we consider changes in children's weight across ages 5 to 13. Because the mean growth pattern across these years evidences curvature, we examined the following quadratic growth model for each child:

$$y_{it} = \boldsymbol{\eta}_{0i} + \boldsymbol{\eta}_{1i}t_{it} + \boldsymbol{\eta}_{2i}t_{it}^2 + \varepsilon_{it}. \tag{7}$$

The interpretation of the highest order coefficient, $\boldsymbol{\eta}_{2i}$ in the present example, is unaffected by the placement of time's origin, whereas the interpretation of lower order terms (i.e., $\boldsymbol{\eta}_{0i}$ and $\boldsymbol{\eta}_{1i}$) is conditional on this placement (Aiken & West, 1991; Cohen, 1978). The intercept and linear components, $\boldsymbol{\eta}_{0i}$ and $\boldsymbol{\eta}_{1i}$, are the predicted value and the instantaneous rate of change, respectively, when time equals zero. The quadratic component, $\boldsymbol{\eta}_{2i}$, indicates acceleration in growth. More specifically, 2* $\boldsymbol{\eta}_{2i}$, the second derivative of Equation 7 with respect to time, is the rate of change in the linear component for a 1-unit change in time which, in the examples presented so far, has been a year. Placing the origin of time at age 5 results in the following loading matrix for the quadratic model:

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \\ 1 & 8 & 64 \end{bmatrix}.$$

Table 3, Model A, presents the results of the quadratic growth model with the origin of time placed at age 5. The overall quadratic growth model adequately represents change in weight over time, $T_{\mathrm{ML}}(6, N = 155) = 26.53$, $p < .001$, CFI = .993, IFI = .994, TLI = .984, RMSEA = .149, 90% CI = .094–.209. The mean intercept value of 39.563 indicates the average weight at age 5 and is very close to the estimate from the linear model based on ages 5 to 9. The mean linear and quadratic components of 6.998 and 0.373, respectively, indicate the average rate at which children are gaining weight at age 5 and the acceleration of that weight gain. However, there is significant variability among children in their weight and rate of change at age 5 (intercept and linear components, respectively) as well as in their acceleration in weight gain across time.

Now consider the impact of several other codings of time presented in Table 3: centering time at age 9 (Model C) and placing the origin of time at age 13 (Model B). The interpretation of Model OP, which uses the quadratic orthogonal polynomial contrast codes commonly found in analysis of variance textbooks for five observations (e.g., Keppel, 1991; Snedecor & Cochran, 1989), is discussed in more detail shortly.

What is the impact of recoding time in the more complicated quadratic model? Equations 4–6, examined first in the context of the linear model, represent general solutions to understanding the impact of recoding time for growth curve models. Consequently, we can use these equations to determine how shifting the model from the intercept coded at age 5 to, for example, age 9 or age 13 affects the variances and covariances of the intercept, linear, and quadratic components as well as the mean vector. The analytically derived parameter estimates and standard errors correspond exactly with the estimated solutions for

[3] To illustrate the unbalanced design using the NLSY child weight data, we conducted several additional analyses in which 20% of the data points were randomly deleted and reestimated the model using Amos 4.01. The estimated models under different codings of time corresponded exactly with the analytic transformations.

Table 3
Quadratic Unconditional Model Estimates and Standard Errors Under Four Time Coding Schemes for Ages 5 to 13

	Model							
	A		B		C		OP	
Parameter	Estimate	*SE*	Estimate	*SE*	Estimate	*SE*	Estimate	*SE*
Variance								
Intercept	33.913	6.818	1,078.68	132.19	274.56	33.56	297.487	34.907
Linear	10.749	2.277	35.968	6.372	13.512	1.694	54.049	6.778
Quadratic	0.154	0.040	0.154	0.040	0.154	0.040	2.462	0.644
Covariance								
Intercept–linear	10.238	3.532	170.796	25.924	52.689	6.760	117.987	14.649
Quadratic–intercept	0.126	0.385	6.431	1.724	0.817	0.726	8.192	3.117
Quadratic–linear	–0.443	0.248	2.019	0.479	0.788	0.204	6.305	1.632
M								
Intercept	39.563	0.489	119.357	2.721	73.489	1.372	76.474	1.409
Linear	6.988	0.322	12.960	0.557	9.974	0.307	19.949	0.614
Quadratic	0.373	0.042	0.373	0.042	0.373	0.042	1.493	0.168
Unique variance								
Age 5	2.942	5.514	2.942	5.514	2.942	5.514	2.942	5.514
Age 7	15.084	3.689	15.084	3.689	15.084	3.689	15.084	3.689
Age 9	44.858	7.150	44.858	7.150	44.858	7.150	44.858	7.150
Age 11	85.200	13.755	85.200	13.755	85.200	13.755	85.200	13.755
Age 13	73.285	34.486	73.285	34.486	73.285	34.486	73.285	34.486
Model $T_{ML}(df = 6)$	26.531		26.531		26.531		26.531	

Note. The T_{ML} is the model test statistic and is asymptotically distributed as a chi-square when the model assumptions are met. OP = orthogonal polynomial. Model loading matrices were as follows for Models A, B, C, and OP, respectively:

$$\boldsymbol{\Lambda}_A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \\ 1 & 8 & 64 \end{bmatrix}, \boldsymbol{\Lambda}_B = \begin{bmatrix} 1 & -8 & 64 \\ 1 & -6 & 36 \\ 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}, \boldsymbol{\Lambda}_C = \begin{bmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}, \text{and } \boldsymbol{\Lambda}_{OP} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}.$$

the models presented in Table 3. The placement of time's origin provides information about individual differences in weight and individual differences in the rate of change in weight growth at specific ages—ages 5, 13, and 9, respectively, in the present example. The mean and variance across individuals of the highest order coefficient for time (quadratic in the present case) are unaffected by different placements of the origin. The choice of where to place the origin of time has to be substantively driven. Because this choice determines that point in time at which individual differences will be examined for the lower order coefficients, the answer to which coding(s) of time to examine in detail lies with the researcher's specific substantive questions of interest.

Conditional models: Predicting growth curve components. We now extend the quadratic growth model and include the mother's weight before pregnancy as a predictor of the growth curve components across children. This involves extending Equation 7 and predicting each of the growth curve components as follows:

$$\eta_{0i} = \mu_{\eta 0} + \gamma_0 M_i + \zeta_{0i},$$

$$\eta_{1i} = \mu_{\eta 1} + \gamma_1 M_i + \zeta_{1i},$$

$$\eta_{2i} = \mu_{\eta 2} + \gamma_2 M_i + \zeta_{2i}.$$

Note that mother's weight (M_i) is centered in all analyses. Table 4, Model A, presents the results of the quadratic growth model with the origin of time placed at age 5, which fits the data adequately, $T_{ML}(8, N = 155) = 26.68$, $p < .01$, CFI = .994, IFI = .994, TLI = .985, RMSEA = .123, 90% CI = .073–.176. Mother's weight is only marginally related to her child's weight at age 5 (the intercept) but is positively related to the rate of weight growth at age 5 (the linear

Table 4
Quadratic Conditional Model Estimates and Standard Errors Under Three Time Coding Schemes for Ages 5 to 13

	Model					
	A		B		C	
Parameter	Estimate	*SE*	Estimate	*SE*	Estimate	*SE*
Residual variance						
Intercept	32.709	6.691	861.401	107.934	233.365	28.404
Linear	9.623	2.170	29.854	5.741	10.539	1.365
Quadratic	0.144	0.039	0.144	0.039	0.144	0.039
Residual covariance						
Intercept–linear	9.403	3.439	134.646	21.986	41.678	5.502
Quadratic–intercept	0.029	0.378	5.087	1.542	0.258	0.667
Quadratic–linear	−0.518	0.243	1.782	0.449	0.632	0.183
Intercept						
Intercepts	39.564	0.483	119.370	2.451	73.489	1.271
Linear	6.987	0.312	12.965	0.522	9.976	0.274
Quadratic	0.374	0.041	0.374	0.041	0.374	0.041
Mother's weight as a predictor of:						
Intercept	0.045	0.023	0.697	0.116	0.304	0.060
Linear	0.048	0.015	0.115	0.025	0.081	0.013
Quadratic	0.004	0.002	0.004	0.002	0.004	0.002
Unique variance						
Age 5	3.221	5.447	3.221	5.447	3.221	5.447
Age 7	15.065	3.677	15.065	3.677	15.065	3.677
Age 9	44.732	7.123	44.732	7.123	44.732	7.123
Age 11	84.470	13.450	84.470	13.450	84.470	13.450
Age 13	76.392	33.427	76.392	33.427	76.392	33.427
Model $T_{\mathrm{ML}}(df = 8)$	26.681		26.681		26.681	

Note. The T_{ML} is the model test statistic and is asymptotically distributed as a chi-square when the model assumptions are met. Mother's weight before pregnancy is centered at the mean. Model loading matrices were as follows for Models A, B, and C, respectively:

$$\boldsymbol{\Lambda}_{\mathrm{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \\ 1 & 8 & 64 \end{bmatrix}, \boldsymbol{\Lambda}_{\mathrm{B}} = \begin{bmatrix} 1 & -8 & 64 \\ 1 & -6 & 36 \\ 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}, \boldsymbol{\Lambda}_{\mathrm{C}} = \begin{bmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}.$$

component), as well as being significantly related to acceleration in weight gain.

Care, as always, must be taken in interpreting the remaining components of the growth curve model. The variances and covariances of the growth curve components across children now reflect residual variability—variability across children that is not accounted for by differences among their mothers' weight. For example, the residual variance of the linear component indicates the variability across children in their rate of weight growth at age 5 not accounted for by their mother's weight.

Equations 4–6 present the general solution of how all parameters of a growth curve model can be determined when the scaling or coding of time is changed. Again, we provide the solution for the model with time centered at age 9 (see Model C, Table 4) and at age 13 (see Model B, Table 4). All solutions can be derived precisely from each other from these equations and correspond exactly with the estimated models. Again, because the interpretation of most model parameters depends on this choice of how to code time, the choice of which model to examine in detail depends on the specific substantive questions of interest.

Fully unbalanced data. Thus far we have presented empirical examples based on balanced data in which all individuals were assessed at the same time points. The interested reader can thus use Table 1 to recreate the presented analyses. However, often individuals may be assessed at different time points and the number of assessments for each individual may vary, resulting in an unbalanced design. The ability to determine results analytically based on recoding time is certainly not limited to balanced data.

As an illustration, individuals i and j may have idiosyncratic loading matrices around a common origin—for example, age 5—resulting in assessment schedules such that $\mathbf{\Lambda}_i \neq \mathbf{\Lambda}_j$. Recoding time by, for example, placing the origin at age 9 simply involves changing each individual's loading matrix in the same manner. Thus $\mathbf{\Lambda}_i^* = \mathbf{\Lambda}_i\mathbf{T}$ and $\mathbf{\Lambda}_j^* = \mathbf{\Lambda}_j\mathbf{T}$, where $\mathbf{T}$ is the same transformation matrix. Consequently $\mathbf{T}^{-1} = (\mathbf{\Lambda}_i^{*\prime}\mathbf{\Lambda}_i^*)^{-1}\,\mathbf{\Lambda}_i^{*\prime}\mathbf{\Lambda}_i = (\mathbf{\Lambda}_j^{*\prime}\mathbf{\Lambda}_j^*)^{-1}\,\mathbf{\Lambda}_i^{*\prime}\mathbf{\Lambda}_j$, and it is clear that the equations for transforming parameter estimates (Equations 4–6) as well as for transforming standard errors (Equations A19–A21) are not dependent on a balanced design (a more detailed argument is presented in the Appendix).

This may be seen as well in a brief empirical example. Over a 2-week interval 2 months into the fall semester, 140 undergraduates enrolled within a longitudinal personality study and completed the Rosenberg Self-Esteem Scale (Rosenberg, 1965) using a 0–8 point response scale (for more complete procedural details, see Biesanz, West, & Graziano, 1998; Biesanz & West, 2000). A total of 136 participants completed another self-esteem assessment an average of 15.12 days later ($SD = 3.95$). Also, an additional self-esteem assessment was available on 117 participants who were present for an in-class testing session at the beginning of the semester. This resulted in an unbalanced design with three assessments available for most, but not all, participants coupled with differing assessment schedules over the semester.

We estimated a linear growth model of self-esteem over the course of the semester using SAS Proc Mixed and maximum likelihood. The origin of time was set at the beginning of the semester, and the units of time were days. On average, participants reported moderately high self-esteem at the beginning of the semester ($\boldsymbol{\mu}_{\eta 0} = 5.90301552$) with a significantly positive average slope ($\boldsymbol{\mu}_{\eta 1} = 0.00327923$). Note that although there were significant individual differences in self-esteem at the beginning of the semester, the variance in slopes across individuals was not significant.

Reestimating the model with the origin of time set near the end of the semester (90 days after the first assessment) resulted in a solution that corresponded exactly with the analytically derived transformation. For example, the estimated mean level of self-esteem 90 days into the semester was $\hat{\boldsymbol{\mu}}_{\eta 0} = 6.198144660$, which corresponds to 6 decimal places with the estimated solution of $(5.90301552 + 90*0.00327923)$. Using the presented equations to transform the initial model resulted in a solution for means, variances, and standard errors—even for nonsignificant effects—that corresponded perfectly with the estimated model.

Examining Default Strategies for Coding Time

Centering time and the use of orthogonal polynomial contrast codes in growth curve models, in particular, have appeared frequently in recent years (e.g., Bates & Labouvie, 1995; Brekke, Long, Nesbitt, & Sobel, 1997; DeGarmo & Forgatch, 1997; Kurdek, 1999). Although there are strong logical arguments for centering predictors in multiple regression (e.g., Aiken & West, 1991) and using orthogonal polynomial contrast codes to conduct trend analyses in the multivariate analysis of variance framework (e.g., Keppel, 1991), their apparent use in growth curve models as default coding strategies requires careful consideration.[4] We discuss centering and orthogonal polynomial codings as default strategies with an emphasis on the interpretation of the resulting solution. Later we illustrate how neither approach maximizes precision or statistical power with respect to predictors of growth curve components.

[4] Concern about multicollinearity in growth curve models is frequently expressed (e.g., Brekke, Long, Nesbitt, & Sobel, 1997; Huttenlocher, Haight, Bryk, Seltzer, & Lyons, 1991; Smith, Landry, & Swank, 2000; Stoolmiller, 1995). Elements in the loading matrix ($\mathbf{\Lambda}$) may be correlated with each other, or the covariances in growth curve components across cases may be substantial given certain codings of time. Changes in multicollinearity from recoding time derives from what Marquardt (1980) termed nonessential multicollinearity: Although the interpretation of lower order terms changes with different codings of time, the essential relationships of the growth curve model parameters do not (see Figure 1). The only real potential concern for growth curve models is that the computational efficiency of estimates produced by different statistical packages may depend somewhat on their specific algorithms and these levels of multicollinearity (e.g., Hoel, 1958; Randall & Rayner, 1990). Regardless, the transformations presented in this article allow researchers to efficiently produce and check parameter estimates for different codings of time and their appropriate standard errors. The apparent issue of multicollinearity does not and should not affect the choice of coding time to examine specific substantive questions.

Centering Time

Centering predictors is often recommended in multiple regression for interpretational reasons. The average score is often a meaningful estimate of an underlying population mean and therefore a logical place to interpret and generalize effects (Aiken & West, 1991; see Wainer, 2000, for a similar argument). However this logic does not generally hold for growth curve models. Different researchers may choose to conduct assessments at varying time points. Consequently, the mean assessment point in one design is unlikely to afford generalization to other research designs. Centering time as a default strategy for coding time places a strong dependence on when observations were made during the growth process. For example, parameter estimates for the centered model would likely be very different if we had modeled data for ages 11 to 18 as opposed to ages 5 to 13. Different observational windows on the growth process would then focus on individual differences at different points in time and would not be fully comparable. The choice(s) of how to code time instead should be made so as to best answer specific substantive questions.

Orthogonal Polynomial Codes

Although the solution from regular polynomial codings of time can be exactly transformed into the solution for orthogonal polynomial coding of time (e.g., see Table 3), the coefficients obtained via orthogonal polynomial contrast codes can be difficult to properly interpret. These interpretational difficulties derive from differences in the scaling of time across components and the meaning of lower order coefficients and can create difficulties in determining the temporal precedence of lower order effects.

The first difficulty in interpreting orthogonal polynomial growth curve models concerns the scaling of time across different growth curve components. As an illustration, consider the transformation in the quadratic growth curve model from when time is centered at age 9 (see Model C in Table 3) to the orthogonal polynomial coding (see Model OP in Table 3). The **T** matrix that produces this particular transformation is illustrated as follows:

$$\mathbf{\Lambda}_{\mathrm{OP}} = \mathbf{\Lambda}_{\mathrm{C}}\mathbf{T}, \text{ where } \mathbf{T} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}. \quad (8)$$

The −2 in the first row transforms the intercept from the predicted value at the center of the observational window (age 9) to the average value across the assessments. In quadratic and higher order polynomial models, these are likely to be different values as evidenced by the mean values for the intercept in Table 3 for the centered and orthogonal polynomial models. The differences in scaling time between these two codings are readily apparent in the 1/2 and the 1/4 on the main diagonal of **T**, which correspond to the linear and quadratic components, respectively. Linear coefficients in the orthogonal polynomial coding are twice as large as in Model C, with the linear variance being 4 times greater. Quadratic coefficients in the orthogonal polynomial coding are 4 times greater than those from Model C, with the quadratic variance component being 16 times greater (see Table 3). This scaling difference has a profound effect on interpretation of the linear and quadratic coefficients. Recall that the unit of time is 1 year for Model C. The orthogonal polynomial model, in contrast, has different units of time for the linear and quadratic components—the unit of time is 2 years for the linear component and 4 years for the quadratic component. Properly interpreting the coefficients presented in Table 3 for the orthogonal polynomial codes requires recognizing these differences in scaling across the components. With the use of standard integer orthogonal polynomial codes (e.g., Keppel, 1991) or orthonormalized codings, the scaling change across the different components of the orthogonal polynomial model depends both on the order of the polynomial model examined and the number of assessments and is not readily apparent.

The second, and more serious, potential difficulty with orthogonal polynomial contrast codes lies with the meaning of the lower order coefficients. As an illustration, under regular polynomial coding such as Models A, B, and C, lower order coefficients in the model are interpreted at the origin of time. In contrast, under orthogonal polynomial coding, all lower order coefficients refer to the average value of the regular polynomial growth components across the observations—what Stoolmiller (1995) referred to as *time-averaged coefficients*—with a potential rescaling of the units of time. In terms of understanding the potential interpretational impact this has for growth curve models, it is worth briefly reexamining what orthogonal polynomial codes accomplish. Orthogonal polynomial codes are independent (uncorrelated and hence orthogonal), rendering growth curve components that are unrelated at the level of each individual

i (e.g., $\boldsymbol{\eta}_{0i}$, $\boldsymbol{\eta}_{1i}$, and $\boldsymbol{\eta}_{2i}$ are uncorrelated intraindividually for person i). However, this does not imply that growth curve components across individuals will be orthogonal. Clearly, as is evident in Table 3, interindividual covariances in growth curve components are still substantial with orthogonal polynomial codes.

The price paid to obtain intraindividual orthogonality is the loss of the ability to specify when lower order coefficients occur. Two cases with an identical average value may differ on when in the course of their respective trajectories they experienced that value. Thus, when one examines interindividual correlates of the intercept (average value), there is implicit variability in the assessment point (i.e., time) across individuals as well. One consequence of this is that determining temporal precedence for lower order terms—for example, with respect to predictors or important outcomes such as a randomized treatment conducted right after the middle observation—may be difficult if not impossible. This, in turn, leads to an inability to infer causality in even ideal experimental circumstances for certain lower order terms (Cook & Campbell, 1979).

One apparent benefit that may arise from centering or the use of orthogonal polynomial codes is that, with available computer programs, estimation of the overall growth curve model may be more readily achieved (e.g., see Raudenbush & Bryk, 2002, chap. 6). Regardless, the choice of which codings of time to examine in detail should be driven by specific substantive questions and not by computational difficulties in analyzing the data. With the use of the formulas presented in this article, the solution for other codings of time can be exactly determined once estimation has converged on a solution.

How Should Time Be Coded?

We have argued against using default coding strategies such as orthogonal polynomial contrast codes. How then should time be coded? Our recommendations are twofold. First, time should be coded to produce parameter estimates that are more easily and readily interpretable (e.g., "regular" polynomial codes such as Models A, B, or C in Tables 3 and 4). Second, because each coding of time can be viewed as providing a detailed snapshot or summary of the growth process at a particular point in time, time should be coded to focus attention and understanding where the primary substantive questions lie. If one is primarily interested in understanding effects, relationships, and individual differences at the beginning of the assessed growth process, then placing the origin of time at the initial assessment will provide that information. Similarly, if it is important to understand effects and relationships at the end of the assessed growth process, then one should place the origin of time at the last assessment.

Presenting and Understanding the Growth Process

It is very likely that no one coding of time will answer all questions that researchers and readers may have. For example, it may be important to understand effects at both initial and final assessments as well as at certain specific points in between. Consequently, we encourage researchers to provide graphs of the growth process. This provides the strong benefit of reducing the dependence of the initial choice of how to code time in order to estimate the model and may provide substantive insights not readily apparent from summary output such as that presented in Table 4.

Growth curve models can be viewed as modeling interactions with time—in the present example, the effect of a variable such as mother's weight on the predicted growth in her child's weight is dependent on another variable (the child's age or time). Because lower order effects represent elements of an interaction with time, these "first order effects do *not* represent a *constant effect* across the range of [time]" (Aiken & West, 1991, p. 102). In the presence of interactions in the context of multiple regression (e.g., X and Z interact in predicting Y), Aiken and West recommended graphing *simple slopes* to help display the nature of the interaction—such as the relationship between X and Y at 1 standard deviation below Z's mean, at Z's mean, and at 1 standard deviation above Z's mean.

We encourage similar procedures in the context of growth curve models. Symbolically, we can express the predicted weight of a child ($\hat{y}_i$) as a function of his or her age (Age_i) and his or her mother's weight (M_i) with the following equations:[5]

[5] The usefulness of graphing a modeled relationship depends entirely on the ability of the model to adequately represent the underlying data. Figures 2–4 present the modeled relationship between child's weight, mother's weight, and the child's age and are thus dependent on the specific chosen model to represent the underlying data. It is also critical to include all terms—significant or not—in the equation if the overall model adequately represents the data.

$$y_{it} = \eta_{0i} + \eta_{1i}Age_{it} + \eta_{2i}Age_{it}^2 + \varepsilon_{it},$$

where

$$\eta_{0i} = \mu_{\eta 0} + \gamma_0 M_i + \zeta_{0i},$$

$$\eta_{1i} = \mu_{\eta 1} + \gamma_1 M_i + \zeta_{1i},$$

$$\eta_{2i} = \mu_{\eta 2} + \gamma_2 M_i + \zeta_{2i}.$$

Substituting the information contained in Model A, Table 4, and reducing the symbolic equations to a single expression results in the following:

$$\hat{y}_{it} = (39.564 + .045M_i) + (6.987 + .048M_i)(Age_{it} - 5) + (0.374 + 0.004M_i)(Age_{it} - 5)^2,$$

or equivalently,

$$\hat{y}_{it} = 13.979 + 3.247\,Age_{it} + .374\,Age_{it}^2 + (-.095 + .008\,Age_{it} + .004\,Age_{it}^2)M_i. \quad (9)$$

Following Aiken and West (1991, p. 68), Figure 2 presents the predicted growth pattern for several different mothers' weights. This represents one graphical summary of Equation 9. However, in the context of growth curve models, it may prove useful to examine and provide several other graphs based on the model expressed in Equation 9 to complement Figure 2. For example, one may be interested in the predictive relationship between mother's weight and children's weight at different ages and how this changes over time. How does mother's weight relate to individual differences among children's weight? As well, one may be interested in examining how mother's weight is related to the rate of change in children's weight growth and how this relationship changes over time.

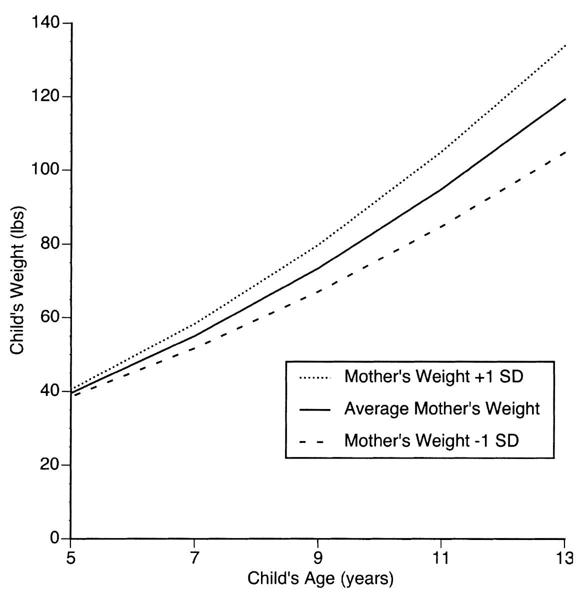

Figure 2. Predicted average child's weight growth as a function of his or her mother's weight before pregnancy. 1 lb = 0.45 kg.

To illustrate, Figure 3 presents the predicted relationship between mother's weight and individual differences in children's weight from ages 5 to 13 coupled with a 95% CI. Each of these graphed points thus represents the relationship (regression coefficient) between mother's weight and the intercept if time had been coded to have its origin at that age. These exact relationships can be determined from either Equation 6 or Equation 9, as the regression coefficient for mother's weight is simply ($-.095 + .008\,Age_{it} + .004\,Age_{it}^2$), which results in the slight quadratic relationship with age observed in Figure 3. Thus, all of the information, including the ability to produce the 95% CI, is contained within the output from a single growth curve model.[6] The graphed value at age 5 (.045) is the relationship between mother's weight and the intercept in Model A in Table 4—where the intercept of time was placed at age 5. Similarly, the graphed values at age 9 (.304) and at age 13 (.697) correspond to the relationships between mother's weight and the intercept for Models C and B in Table 4, respectively. Each point on this graph represents a simple slope—the relationship between mother's weight and individual differences in children's weight at a given age.

Presenting the results of the growth curve analysis as in Figure 3 may provide insights that are not readily apparent from the model output contained in Table 4. For example, we readily observe that although mother's weight is not significantly related to individual differences in children's weight at age 5, this predictive relationship increases steadily through age 13 and is significant shortly after age 5.

The substantive questions of import may focus on rates of change as well. Figure 4 graphs the rela-

[6] An additional Appendix containing a SAS Proc IML program using the data from Model A in Table 4 to produce the data graphed in Figures 3 and 4 is available both on the Web at dx.doi.org/10.1037/1082-989X.9.1.30.supp. and from Jeremy C. Biesanz for 2 years after publication.

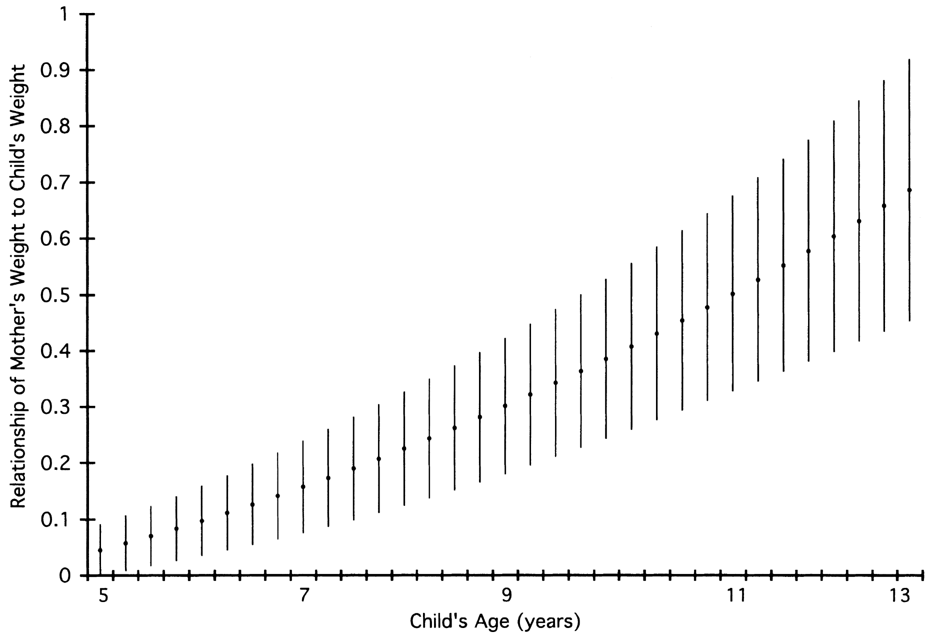

Figure 3. Effect of mother's weight on child's weight from age 5 to age 13 with a 95% confidence interval (vertical bars). The graphed value is the relationship between mother's weight and the intercept for that child's age coded as the origin of time.

tionship between mother's weight and the linear component for different ages. These points represent the tangents to the curve—the instantaneous rate of change—presented in Figure 3. This line can be determined directly from the first derivative of Equation 9 as a function of age, which results in a regression coefficient for mother's weight as $(.008 + .008\,Age_{it})$. Note that the graphed values at age 5 (.048) and at age 13 (.115) correspond identically to the relationship between mother's weight and the linear component presented in Table 4 (Models A and B, respectively) as well as to the first derivative of Equation 9 if more significant digits are carried through the calculations. Through visual inspection alone, it is apparent that mother's weight is significantly related to the rate of weight growth for children during the observed age range of 5 to 13 and that this relationship is increasing over time.[7]

Obtaining these graphs is remarkably simple. More important, providing graphs and examining effects that correspond to the specific research questions of interest across the observational window reduce the dependence on how time was originally coded. Graphing and examining effects across the observational window may also provide clearer insights and understanding of the modeled growth process.

Precision of Estimates and Power Analyses

In determining a choice of coding time, it would seem natural to consider the statistical power that would result for different choices. It is clear from Figure 3 that the relationship between mother's and child's weight is different at age 5 than at age 13 and that their respective standard errors differ as well. The statistical power to detect the relationship between a mother's and her child's weight thus depends on the age of the child. We note that any difference in statistical power as a function of recoding time reflects a change in the question asked—the relationship between mother's and child's weight is a fundamentally different question at age 5 than at age 13 if children are growing at different rates. Nonetheless, considering how statistical power and the precision of esti-

[7] The slope of the line in Figure 4 represents the relationship between mother's weight and the quadratic component. Recall that 2 times the quadratic component represents the acceleration in the growth process. Thus half of the slope in Figure 4 represents the relationship between mother's weight and the quadratic component—that is, .008/2 = .004, the relationship between mother's weight and the quadratic component presented in Table 4.

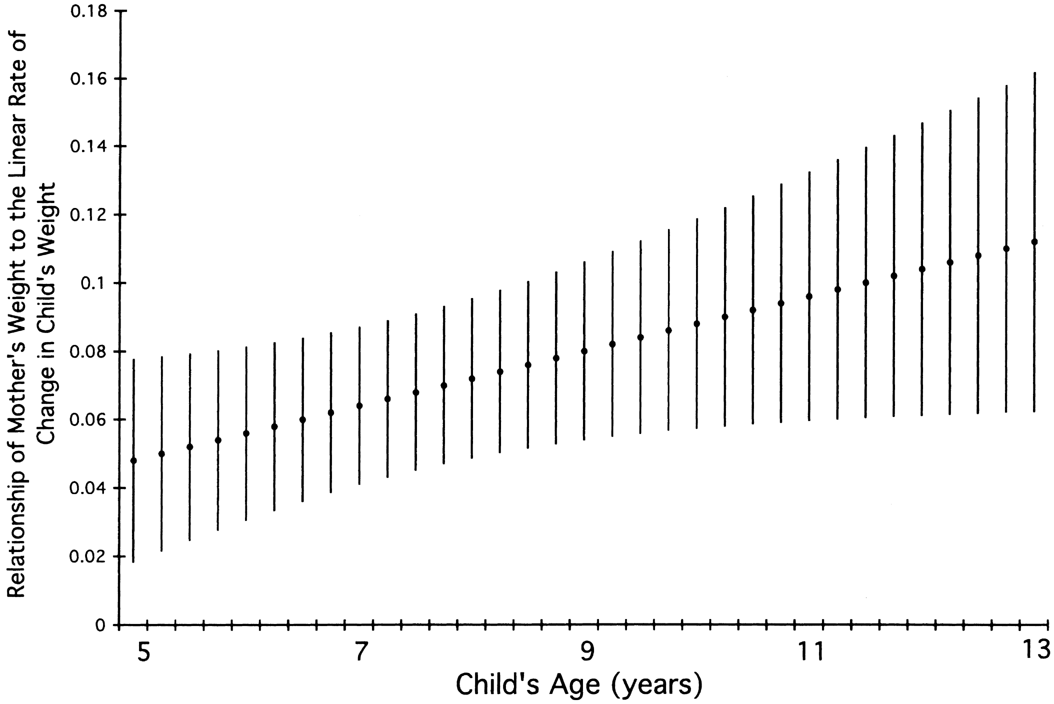

Figure 4. Effect of mother's weight on the instantaneous linear rate of change in child's weight at different ages with a 95% confidence interval (vertical bars). The graphed value is the relationship between mother's weight and the linear component for that child's age coded as the origin of time.

mates change as a function of coding time highlights the importance of understanding the overall growth process for interpreting effects.

We focus our attention on understanding the relationship between a mother's weight and her child's weight and linear rate of change in weight that correspond to the graphs presented in Figures 3 and 4. Regardless of how time is coded, the statistical power to detect the relationship between mother's weight and the highest order term—quadratic rate of change—remains constant. These predicted relationships are exactly determined from the summary equation (Equation 9). However, for the relationship between mother's and child's weight, why is it that the precision of estimates (i.e., the tightness of the CI) is best at age 5 and increases with age? Similarly, for the relationship between mother's weight and the linear rate of change in child's weight, why are estimates most precise just after age 8? Although Equations A18 and A20 describe the mechanics of how the variance of each estimate changes with different codings of time, they do not provide an explanation.

An understanding of these effects can be obtained from considering the following consistent estimate of the covariance matrix of the vector of regression coefficients (Verbeke & Molenberghs, 2000; Zeger, Liang, & Albert, 1988):[8]

$$\widehat{\mathrm{COV}}(\boldsymbol{\Gamma}) = \left\{ \sum_{i=1}^{n} [X_i' \boldsymbol{\Lambda}' (\boldsymbol{\Lambda\Psi\Lambda}' + \boldsymbol{\Theta}_{\varepsilon\varepsilon})^{-1} \boldsymbol{\Lambda} X_i] \right\}^{-1},$$

$$\widehat{\mathrm{COV}}(\boldsymbol{\Gamma}^*) = \left\{ \sum_{i=1}^{n} [X_i' \mathbf{T}' \boldsymbol{\Lambda}' (\boldsymbol{\Lambda\Psi\Lambda}' + \boldsymbol{\Theta}_{\varepsilon\varepsilon})^{-1} \boldsymbol{\Lambda} \mathbf{T} X_i] \right\}^{-1}. \quad (10)$$

The residual variance contained in the inner inverse $(\boldsymbol{\Lambda\Psi\Lambda}' + \boldsymbol{\Theta}_{\varepsilon\varepsilon})$ remains unchanged after recoding time. Because recoding time simply reweights this residual variance, we can understand the variance of the regression coefficients—and hence their preci-

[8] Finite samples result in a larger estimated covariance matrix because of sampling variability derived from estimating the variance components (see McCulloch & Searle, 2001); nonetheless, this estimate is useful for understanding changes in the variance of regression coefficients. For all graphs, however, we present the actual estimated variances.

sion—as a complex function of the three components contained within this inner inverse. First, the precision of the regression coefficient estimates will be impacted by the heterogeneity among the uniquenesses, if any, of the assessments. Within the current data set such heterogeneity—differences in the unique variances across assessments—has no immediately apparent impact. Second, the timing of assessments, conveyed by the $\mathbf{\Lambda}$ matrix, impacts the precision of the regression coefficients. Estimates of effects at the center of the assessment schedule will generally be more precise. Third, the residual variance of the growth curve components is closely tied to the precision of the regression coefficients. We consider these latter two effects in detail.

Figure 3 shows that the precision of the relationship between the mother's weight and her child's weight decreases with age. As is seen in Figure 5A, the increase in the variance of the intercept's regression coefficient corresponds almost perfectly with the monotonic increase in the residual variance in the intercept (child's weight) with age. Note that the graphed values in Figure 5A at ages 5, 9, and 13 correspond to the estimated values from Models A, C, and B, respectively, in Table 4, for the intercept residual variance and the square of the standard error of mother's weight as a predictor of the intercept (i.e., the intercept regression coefficient variance). This strong correspondence between the variance of the residual growth curve components and the variance of the regression coefficient is seen as well in Figure 5B for the linear rate of change, whose graphed values correspond as well to Table 4. Examining the linear rate of change reveals the influence of the assessment schedule in the precision of regression coefficient estimates. The linear residual variance is smallest at approximately 6.8 years of age. In contrast, the regression coefficient is most precise (has the lowest variance) at approximately 7.4 years of age—closer to the middle of the assessment schedule.

Combining regression coefficient estimates with their variance allows us to explore how the estimated statistical power for the intercept and linear component changes throughout the range of observed data. Figures 6A and 6B present the estimated statistical power for this study to detect the relationship between the mother's weight and her child's weight and linear rate of change in weight, respectively, at different child's ages. Plotted concurrently is the chi-square noncentrality parameter used to determine statistical power. The noncentrality parameter is the Wald statistic, calculated by squaring the regression coefficient and then dividing by its variance (Bollen, 1989).

The statistical power to detect the relationship between mother's and child's weight and linear rate of change is substantial for both effects after age 6. Statistical power peaks at approximately age 12.4 and 8.4 for the intercept and linear components, respectively. Although the regression coefficients for both of these effects are increasing monotonically throughout the assessment window, the variance of the regression coefficient increases faster after these ages. Considering Figures 6A and 6B together, it is clear that no one coding of time will result in maximal precision or statistical power. Orthogonal polynomial coding does not offer a solution either—the noncentrality parameters for orthogonal polynomial coding are 29.87 and 38.05 for the intercept and linear components, respectively, and below their maximum values of 34.96 and 39.2.

It is clear from Figures 2 and 3 that mother's weight strongly predicts individual differences among child's weight and that this relationship increases from age 5 to age 13. The precision of that relationship depends primarily on the individual differences not accounted for by mother's weight—the residual variance component—and on the timing of the assessments. This highlights the importance of study design—when to gather assessments—and the specification of a priori and focused hypotheses. To illustrate, let us suppose that we ask the seemingly simple question of whether a mother's weight predicts her child's weight. In the present example, we observe from Table 4 that the mother's weight is not significantly related to her child's weight at age 5, an effect examined with relatively low power, but that statistical power increases dramatically after that age and is significant at both age 9 and age 13. Although the question seems simple, the answer is not as long as children are growing at different rates—regardless of whether mother's weight predicts the quadratic growth component. Any answer to that question must be qualified by considering the age of the child or, for the case of orthogonal polynomials, by averaging across ages 5 to 13, and must be driven by specific hypotheses.

Examining only one coding of time by itself will not provide a full and complete picture of the growth process, nor will any one coding of time provide maximal statistical power or precision for all estimates. Understanding the growth process and changes in the precision of estimates and statistical power requires understanding how individual differences are

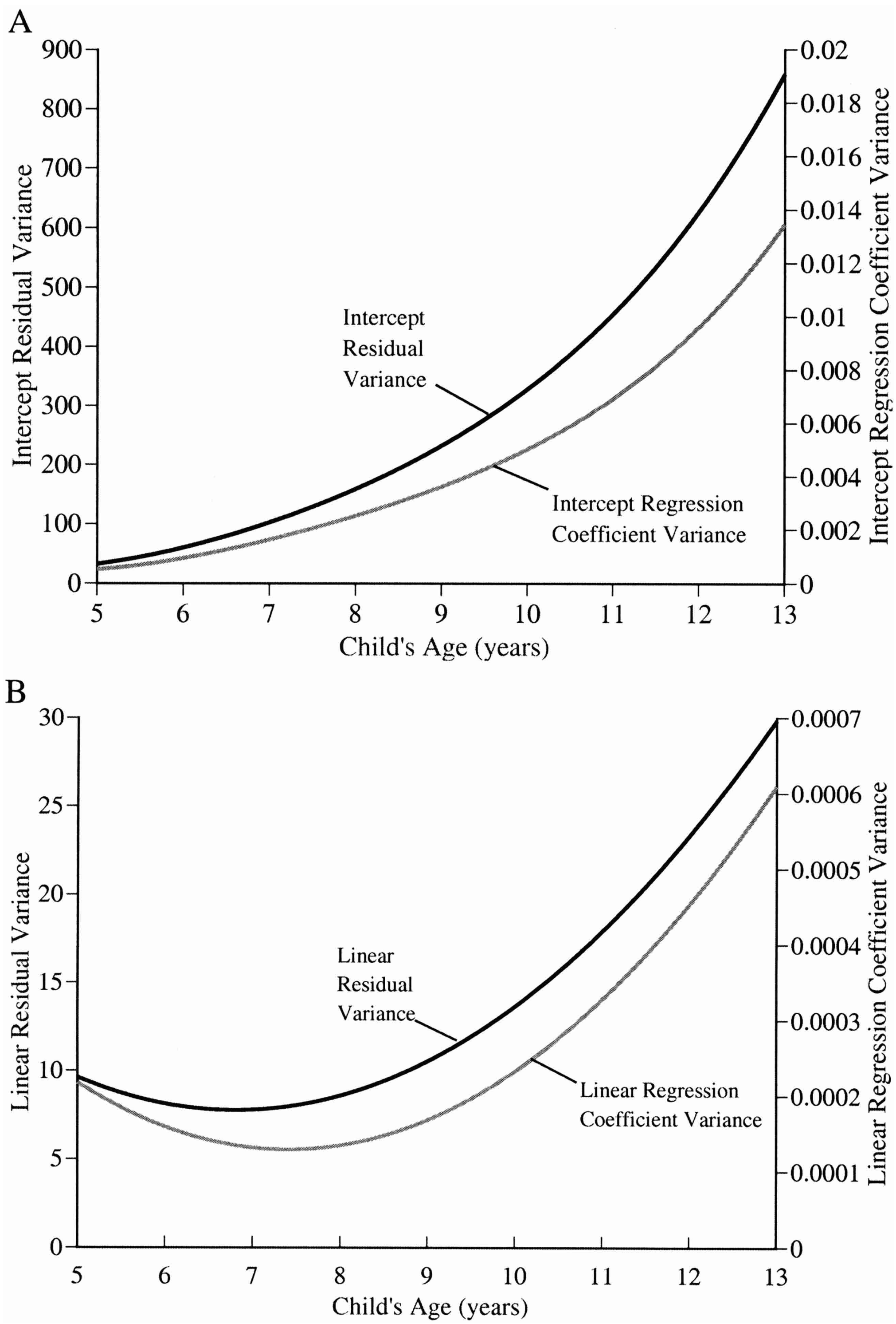

Figure 5. Residual variance in child's weight after accounting for mother's weight and the variance of that regression relationship as a function of child's age. A: Intercept (the relationship between mother's and child's weight). B: Linear component (the relationship between mother's weight and linear rate of change in the child's weight).

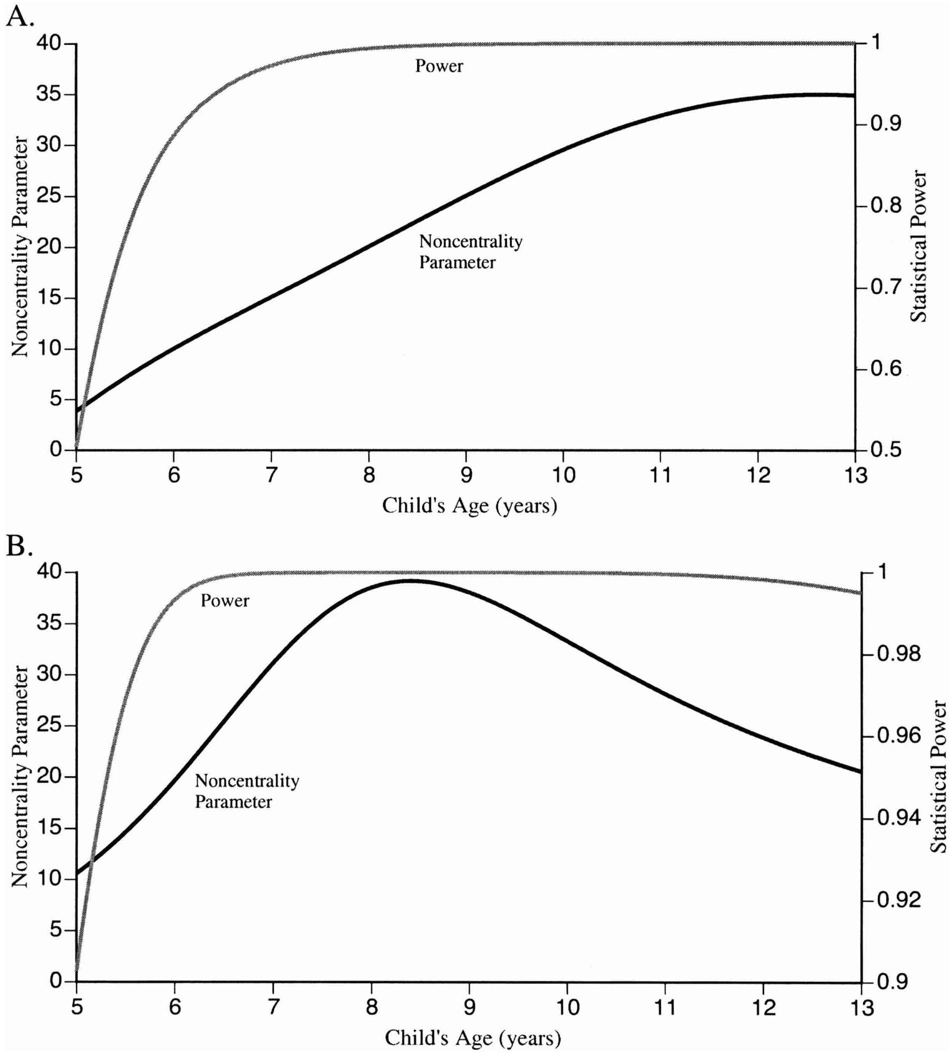

Figure 6. Estimated statistical power and chi-square noncentrality parameter based on N = 155 and α = .05 for detecting the relationship between mother's weight and the intercept (A) and linear component (B) effects for that child's age coded as the origin of time.

changing across time. Graphing effects and individual differences (variances) across time provides one route to achieving that understanding.

Summary and Conclusions

Identifying and estimating growth curve models requires choosing how to code time, and the interpretation of parameters in the resulting model depends critically on this choice. For alternative codings of time, however, parameter estimates and their standard errors can be directly determined analytically. Consequently, the choice of where to examine individual differences on the construct of interest and on components of change depends on the specific substantive questions and the research design. Furthermore, instead of presenting only one specific coding of time, which provides detailed information on a quick snapshot or brief summary of the growth process, we encourage researchers to graphically present the growth process in manners that correspond to the primary substantive questions. For example, researchers should present traditional graphs such as Figure 2 coupled with graphs of the simple effects of substantively important lower order coefficients in the growth model across the observational window. These graphs represent a more complete analysis of the growth process implied by the model coupled with regions of significance of effects. Presenting these relationships and the nature of growth implied by a model may help in both interpreting and conveying the rich and useful information contained in growth curve models and provides opportunities to gain new insights into the growth process.

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Appendix

Transforming Parameter Estimates and Standard Errors in Growth Curve Models

Let $\mathbf{y}$ be a matrix of observed responses at different points in time for every case, $\boldsymbol{\Lambda}$ the loading matrix that specifies the functional form of growth, $\boldsymbol{\eta}$ the matrix of unobserved growth coefficients across cases, and $\boldsymbol{\varepsilon}$ the matrix corresponding to disturbances at each assessment for each case. For ease of presentation, we initially assume the balanced case without any loss of generality and later discuss the unbalanced case:

$$\mathbf{y} = \boldsymbol{\Lambda\eta} + \boldsymbol{\varepsilon}. \tag{A1}$$

The predicted mean value across cases at each assessment ($\boldsymbol{\mu}_{\mathbf{y}}$) is thus related to the mean growth coefficient across cases ($\boldsymbol{\mu}_{\boldsymbol{\eta}}$) as follows:

$$\boldsymbol{\mu}_{\mathbf{y}} = \boldsymbol{\Lambda\mu}_{\boldsymbol{\eta}}. \tag{A2}$$

Equation A1 implies a covariance matrix across cases for the different assessments:

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \boldsymbol{\Lambda\Sigma}_{\boldsymbol{\eta\eta}}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}}. \tag{A3}$$

Suppose T is a nonsingular transformation matrix such that:

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}\mathbf{T},$$

and (A4)

$$\boldsymbol{\eta}^* = \mathbf{T}^{-1}\boldsymbol{\eta}.$$

It then follows that

$$\mathbf{y} = \boldsymbol{\Lambda}^*\boldsymbol{\eta}^* + \boldsymbol{\varepsilon}, \tag{A5}$$

$$\boldsymbol{\mu}_{\mathbf{y}} = \boldsymbol{\Lambda}^*\boldsymbol{\mu}_{\boldsymbol{\eta}^*}, \tag{A6}$$

and

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \boldsymbol{\Lambda}^*\boldsymbol{\Sigma}_{\boldsymbol{\eta}^*\boldsymbol{\eta}^*}\boldsymbol{\Lambda}^{*\prime} + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}}. \tag{A7}$$

Equations A5 to A7 presume that the loss function used to estimate the model is unaffected by the scaling transformation in Equation A4. Scale-free estimators such as maximum likelihood satisfy this condition, and, consequently from Equations A3 and A7,

$$\boldsymbol{\Lambda\Sigma}_{\boldsymbol{\eta\eta}}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}} = \boldsymbol{\Lambda}^*\boldsymbol{\Sigma}_{\boldsymbol{\eta}^*\boldsymbol{\eta}^*}\boldsymbol{\Lambda}^{*\prime} + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}}.$$

Because $\boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}}$ drops out, the form and restrictions on uniquenesses are immaterial to scaling transformations defined by Equation A4:

$$\boldsymbol{\Lambda\Sigma}_{\boldsymbol{\eta\eta}}\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}^*\boldsymbol{\Sigma}_{\boldsymbol{\eta}^*\boldsymbol{\eta}^*}\boldsymbol{\Lambda}^{*\prime},$$

and

$$\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda\Sigma}_{\boldsymbol{\eta\eta}}\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}^*\boldsymbol{\Sigma}_{\boldsymbol{\eta}^*\boldsymbol{\eta}^*}\boldsymbol{\Lambda}^{*\prime}.$$

Assuming that ($\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}^*$) is nonsingular (i.e., that the columns in the loading matrix in the model are not perfectly multicollinear),

$$\boldsymbol{\Sigma}_{\boldsymbol{\eta}^*\boldsymbol{\eta}^*} = (\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}^*)^{-1}\,\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda\Sigma}_{\boldsymbol{\eta\eta}}\boldsymbol{\Lambda}'\boldsymbol{\Lambda}^*(\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}^*)^{-1}. \tag{A8}$$

From Equations A2 and A6,

$$\boldsymbol{\Lambda\mu}_{\boldsymbol{\eta}} = \boldsymbol{\Lambda}^*\boldsymbol{\mu}_{\boldsymbol{\eta}^*}, \tag{A9}$$

and

$$\boldsymbol{\mu}_{\boldsymbol{\eta}^*} = (\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}^*)^{-1}\,\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda\mu}_{\boldsymbol{\eta}}.$$

Because for a particular case, $\boldsymbol{\eta} = \boldsymbol{\mu}_{\boldsymbol{\eta}} + \boldsymbol{\zeta}$, consequently $\boldsymbol{\Sigma}_{\boldsymbol{\eta\eta}} = \boldsymbol{\Psi}$ and Equations A8 and A9 correspond to Equations 4 and 5, respectively.

Conditional Growth Model

Extending this model to include predictors of growth curve components where (X) is the set of predictors and ($\boldsymbol{\Gamma}$) is the relationship between X and $\boldsymbol{\eta}$ results, for a single case, in the following:

$$\boldsymbol{\eta} = \boldsymbol{\mu}_{0\boldsymbol{\eta}} + \boldsymbol{\Gamma}X + \boldsymbol{\zeta}, \tag{A10}$$

and

$$\boldsymbol{\eta}^* = \boldsymbol{\mu}_{0\boldsymbol{\eta}^*} + \boldsymbol{\Gamma}^*X + \boldsymbol{\zeta}^*. \tag{A11}$$

Note that $\boldsymbol{\mu}_{0\boldsymbol{\eta}}$ and $\boldsymbol{\mu}_{0\boldsymbol{\eta}^*}$ are now intercept vectors that express the value of the growth curve components when the predictors are all zero. Consequently, the covariance among the **y**s and between X and **y** and the means of **y** may be expressed as follows:

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \boldsymbol{\Lambda}(\boldsymbol{\Gamma}\boldsymbol{\Phi}\boldsymbol{\Gamma}' + \boldsymbol{\Psi})\boldsymbol{\Lambda}' + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}} = \boldsymbol{\Lambda}^*(\boldsymbol{\Gamma}^*\boldsymbol{\Phi}\boldsymbol{\Gamma}^{*\prime} + \boldsymbol{\Psi}^*)\boldsymbol{\Lambda}^{*\prime} + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}}, \tag{A12}$$

$$\boldsymbol{\Sigma}_{\mathbf{xy}} = \boldsymbol{\Phi}\boldsymbol{\Gamma}'\boldsymbol{\Lambda}' = \boldsymbol{\Phi}\boldsymbol{\Gamma}^{*\prime}\boldsymbol{\Lambda}^{*\prime}, \tag{A13}$$

and

$$\boldsymbol{\mu}_{\mathbf{y}} = \boldsymbol{\Lambda}(\boldsymbol{\mu}_{0\boldsymbol{\eta}} + \boldsymbol{\Gamma}X) = \boldsymbol{\Lambda}^*(\boldsymbol{\mu}_{0\boldsymbol{\eta}^*} + \boldsymbol{\Gamma}^*X). \tag{A14}$$

Solving Equations A12, A13, and A14 for $\boldsymbol{\Psi}^*$, Γ^*, and $\boldsymbol{\mu}_{0\boldsymbol{\eta}^*}$ results in the following:

$$\boldsymbol{\Psi}^* = \mathbf{T}^{-1}\boldsymbol{\Psi}\mathbf{T}^{-1\prime}, \tag{A15}$$

$$\boldsymbol{\mu}_{0\boldsymbol{\eta}^*} = \mathbf{T}^{-1}\,\boldsymbol{\mu}_{0\boldsymbol{\eta}}, \tag{A16}$$

and

$$\boldsymbol{\Gamma}^* = \mathbf{T}^{-1}\boldsymbol{\Gamma}, \tag{A17}$$

where $\mathbf{T}^{-1} = (\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}^*)^{-1}\boldsymbol{\Lambda}^{*\prime}\boldsymbol{\Lambda}$ and $\mathbf{T}$ is as defined in Equation A4.

Although this represents the impact of transformations in the conditional growth model, note that Equations A15 and A16 correspond, respectively, to Equations A8 and A9 from the unconditional growth model.

Transforming Standard Errors in Growth Curve Models

Just as all parameter estimates in a growth curve model are exactly determined from a transformation such as recoding time, so are the standard errors of transformed parameter estimates. As an illustration, let $\hat{\boldsymbol{\theta}}$ be a $p \times 1$ vector of latent growth curve model parameter estimates given a particular coding of time with asymptotic covariance matrix ACOV $(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}}')$, and let $\hat{\boldsymbol{\theta}}^*$ be the vector of model parameter estimates under a transformation of time such as those described earlier in the Appendix. Let $\mathbf{J}_{\hat{\boldsymbol{\theta}}^* \to \hat{\boldsymbol{\theta}}}$ be the Jacobian matrix of partial derivatives of the transformation of $\hat{\boldsymbol{\theta}}^*$ to $\hat{\boldsymbol{\theta}}$ (see Searle, 1982, section 12.10.d):

$$\mathbf{J}_{\hat{\boldsymbol{\theta}}^* \to \hat{\boldsymbol{\theta}}} = \left(\frac{\partial \hat{\boldsymbol{\theta}}^*}{\partial \hat{\boldsymbol{\theta}}}\right)' = \left[\frac{\partial \hat{\boldsymbol{\theta}}_i^*}{\partial \hat{\boldsymbol{\theta}}_j}\right]_{ij} \quad \text{for } i,j = 1, \cdots, p.$$

Suppose that for a given parameter estimate, say $\hat{\boldsymbol{\theta}}_j$, we wish to determine its standard error after a time coding transformation results in a transformed estimate $\hat{\boldsymbol{\theta}}_j^*$. From the multivariate delta method for distributions of transformed variables (Bishop, Fienberg, & Holland, 1975, section 14.6; see also Cudeck & O'Dell, 1994, for a parallel example),

$$\mathrm{AVAR}(\hat{\boldsymbol{\theta}}_j^*) = (\mathbf{J}_{[j]})'\mathrm{ACOV}(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}}')(\mathbf{J}_{[j]}), \tag{A18}$$

where $\mathbf{J}_{[j]}$ is the jth column of $\mathbf{J}_{\hat{\boldsymbol{\theta}}^* \to \hat{\boldsymbol{\theta}}}$. Consequently, the asymptomatic standard error of $\hat{\boldsymbol{\theta}}_j^*$ is $\sqrt{\mathrm{AVAR}(\hat{\boldsymbol{\theta}}_j^*)}$. Note that although we are interested in determing the standard error of $\hat{\boldsymbol{\theta}}_j^*$ when transformed from $\hat{\boldsymbol{\theta}}_j$, obtaining the appropriate partial derivatives requires using the Jacobian matrix of the transformation from $\hat{\boldsymbol{\theta}}_j^*$ to $\hat{\boldsymbol{\theta}}_j$.

Applying this technique to growth curve models necessitates determining analytically the Jacobian matrix $\mathbf{J}_{\hat{\boldsymbol{\theta}}^* \to \hat{\boldsymbol{\theta}}}$ for the different sets of parameter estimates of interest. Below we derive the relevant Jacobian transformation matrices for $\boldsymbol{\mu}$, $\boldsymbol{\Gamma}$, and $\boldsymbol{\Psi}$ and then present a brief example illustrating how to compute these transformed standard errors.

For the vector of estimated latent growth curve means (or intercepts in the case of the conditional model),

$$\mathbf{J}_{\boldsymbol{\mu}^* \to \boldsymbol{\mu}} = \left(\frac{\partial \boldsymbol{\mu}^*}{\partial \boldsymbol{\mu}}\right)' = \left(\frac{\partial \mathbf{T}^{-1}\boldsymbol{\mu}}{\partial \boldsymbol{\mu}}\right)' = \mathbf{T}^{-1\prime}. \tag{A19}$$

For the vector of regression coefficients of predictors of growth curve components, that is, vec($\boldsymbol{\Gamma}$),

$$\begin{aligned}\mathbf{J}_{\mathrm{vec}(\boldsymbol{\Gamma}^*) \to \mathrm{vec}(\boldsymbol{\Gamma})} &= \left[\frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}^*)}{\partial \mathrm{vec}(\boldsymbol{\Gamma})}\right]' \\ &= \left[\frac{\partial(\mathbf{I} \otimes \mathbf{T}^{-1})\mathrm{vec}(\boldsymbol{\Gamma})}{\partial \mathrm{vec}(\boldsymbol{\Gamma})}\right]' \\ &= (\mathbf{I} \otimes \mathbf{T}^{-1})'. \end{aligned} \tag{A20}$$

Note that the order of $\mathbf{I}$ equals the number of columns of $\boldsymbol{\Gamma}$ (i.e., the number of predictors). The vec() operator stacks the columns of a matrix one under each other and thus turns a matrix into a vector (see Searle, 1982, section 12.9.a).

For the vector of the covariance matrix of latent growth curve components, that is, vec($\boldsymbol{\Psi}$), that includes redundant elements,

$$\begin{aligned}\mathbf{J}_{\mathrm{vec}(\boldsymbol{\Psi}^*) \to \mathrm{vec}(\boldsymbol{\Psi})} &= \left[\frac{\partial \mathrm{vec}(\boldsymbol{\Psi}^*)}{\partial \mathrm{vec}(\boldsymbol{\Psi})}\right]' \\ &= \left[\frac{\partial(\mathbf{T}^{-1} \otimes \mathbf{T}^{-1})\mathrm{vec}(\boldsymbol{\Psi})}{\partial \mathrm{vec}(\boldsymbol{\Psi})}\right]' \\ &= (\mathbf{T}^{-1} \otimes \mathbf{T}^{-1})'. \end{aligned} \tag{A21}$$

Note that this transformation applies for unconditional as well as conditional models.

We present a brief illustration on how to obtain transformed standard errors for elements of the covariance matrix of latent growth components. Consider the linear growth model in Table 2 and the transformation from Model A to Model B where

$$\text{Model A } \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}, \text{ and Model B } \boldsymbol{\Lambda}^* = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}.$$

Define $\boldsymbol{\Psi}_\alpha$ as the variance of latent intercepts, $\boldsymbol{\Psi}_\beta$ as the variance of the latent slopes, and $\boldsymbol{\Psi}_{\alpha\beta}$ as the covariance between intercept and slope. The estimated covariance matrix of the parameter estimates of $\boldsymbol{\Psi}$ from Amos 4.01 for Model A is as follows:

(Appendix continues)

$$\text{ACOV}(\boldsymbol{\Psi}, \boldsymbol{\Psi}) = \begin{array}{c} \boldsymbol{\Psi}_\alpha \\ \boldsymbol{\Psi}_{\alpha\beta} \\ \boldsymbol{\Psi}_{\alpha\beta} \\ \boldsymbol{\Psi}_\beta \end{array} \begin{bmatrix} 37.73111857 & -4.44960714 & -4.44960714 & 4.55185580 \\ -4.44960714 & 7.16164762 & 7.16164762 & -.12705333 \\ -4.44960714 & 7.16164762 & 7.16164762 & -.12705333 \\ 4.55185580 & -.12705333 & -.12705333 & 2.16180421 \end{bmatrix}.$$

Note the necessary redundant elements in the covariance matrix of parameter estimates for cov($\boldsymbol{\Psi}$,$\boldsymbol{\Psi}$) to conform with vec($\boldsymbol{\Psi}$). The square root of the elements on the main diagonal corresponds to the standard errors for Model A presented in Table 2.

$$\mathbf{J}_{\text{vec}(\boldsymbol{\Psi}^*)\to\text{vec}(\boldsymbol{\Psi})} = (\mathbf{T}^{-1} \otimes \mathbf{T}^{-1})' =$$

$$\left[\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\right]' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 16 & 4 & 4 & 1 \end{bmatrix}.$$

To estimate avar($\boldsymbol{\Psi}^*_\alpha$) requires extracting its corresponding column from $\mathbf{J}_{\text{vec}(\boldsymbol{\Psi}^*)\to\text{vec}(\boldsymbol{\Psi})}$ and pre- and postmultiplying acov(Ψ, Ψ) by that column:

$$\text{AVAR}(\boldsymbol{\Psi}^*_\alpha) =$$

$$[1\ 4\ 4\ 16]\ \text{ACOV}(\boldsymbol{\Psi}, \boldsymbol{\Psi}) \begin{bmatrix} 1 \\ 4 \\ 4 \\ 16 \end{bmatrix} = 1091.438463.$$

The standard error of $\boldsymbol{\Psi}^*_\alpha$ is consequently $\sqrt{1091.438463} = 33.03693$, which corresponds exactly to the standard error in Table 2, Model B. For producing the standard errors for the covariance between intercept and slope, either the second or third columns of $\mathbf{J}_{\text{vec}(\boldsymbol{\Psi}^*)\to\text{vec}(\boldsymbol{\Psi})}$ will suffice. Finally, because the last column of $\mathbf{J}_{\text{vec}(\boldsymbol{\Psi}^*)\to\text{vec}(\boldsymbol{\Psi})}$ corresponds to the variance of the slope, it is apparent that its standard error is unaffected by the transformation in the location of the intercept of time.

Transforming Parameters in the Unbalanced Design

Thus far we have been assuming a balanced design in which each individual is assessed at exactly the same points in time. However, it is common for individuals to have assessments at slightly different points in time and for some individuals to miss assessments. This may result in an unbalanced design such that for persons k and l, $\boldsymbol{\Lambda}_k \neq \boldsymbol{\Lambda}_l$. As we demonstrate, when a scale-free estimator such as maximum likelihood is used to estimate the overall model, having an unbalanced design has no impact on transformations of the resulting solution due to recoding time.

Even though in an unbalanced design individuals may be assessed at differing points in time, the same transformation is applied to each individual. Consequently, which individual's loading matrix is used for recoding time is immaterial, as can be seen below:

$$(\boldsymbol{\Lambda}^{*\prime}_k \boldsymbol{\Lambda}^*_k)^{-1}\, \boldsymbol{\Lambda}^{*\prime}_k \boldsymbol{\Lambda}_k = (\boldsymbol{\Lambda}^{*\prime}_l \boldsymbol{\Lambda}^*_l)^{-1} \boldsymbol{\Lambda}^{*\prime}_l \boldsymbol{\Lambda}_l = \mathbf{T}^{-1}. \tag{A22}$$

Furthermore, as in the balanced design, the observed values at a specific set of points in time (e.g., $\boldsymbol{\Lambda}_k$), the population mean values, and the covariance across assessments at those points in time are unaffected by the recoding of time:

$$\mathbf{y}_k = \boldsymbol{\Lambda}_k \boldsymbol{\eta}_k + \boldsymbol{\varepsilon} = \boldsymbol{\Lambda}^*_k \boldsymbol{\eta}^*_k + \boldsymbol{\varepsilon}, \tag{A23}$$

$$\boldsymbol{\mu}_{y_k} = \boldsymbol{\Lambda}_k \boldsymbol{\mu}_{\boldsymbol{\eta}} = \boldsymbol{\Lambda}^*_k \boldsymbol{\mu}_{\boldsymbol{\eta}^*}, \tag{A24}$$

and

$$\boldsymbol{\Sigma}_{y_k, y_k} = \boldsymbol{\Lambda}_k \boldsymbol{\Psi} \boldsymbol{\Lambda}'_k + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}} = \boldsymbol{\Lambda}^*_k \boldsymbol{\Psi}^* \boldsymbol{\Lambda}^{*\prime}_k + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon\varepsilon}}. \tag{A25}$$

With unbalanced data, the log of the likelihood for the sample for a given solution is (e.g., see Jennrich & Schluchter, 1986) as follows:

$$\ln L = -1/2 \sum_{k=1}^{N} [p_k \ln(2\pi) + \ln|\Sigma_{y_k y_k}| + (\mathbf{y}_k - \mu_{y_k})' \Sigma^{-1}{}_{y_k y_k} (y_k - \mu_{y_k})],$$

where p_k is the number of assessments for the kth individual. Raw or direct maximum likelihood seeks to maximize this likelihood using all available data. What is apparent from this likelihood function is that even in the unbalanced design, the likelihood function is unaffected by the recoding of time, so long as the same recoding is applied to the entire sample. Consequently, all of the results presented here apply equally to balanced and unbalanced designs.

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